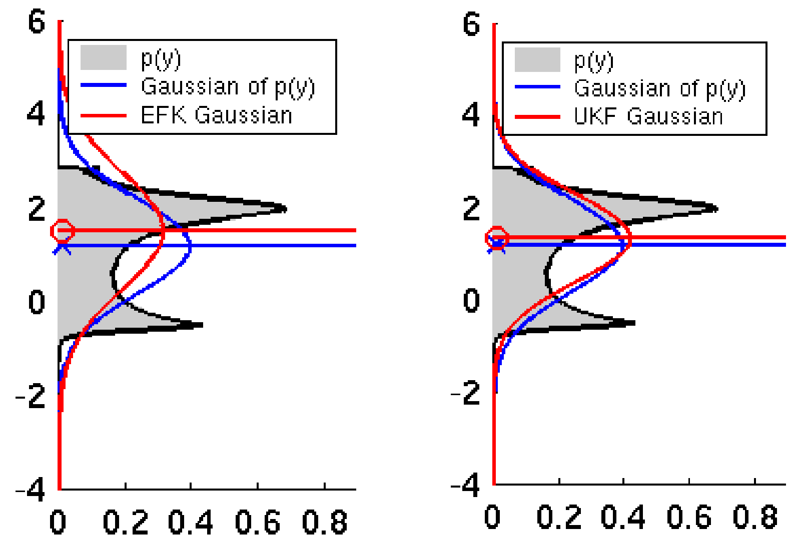
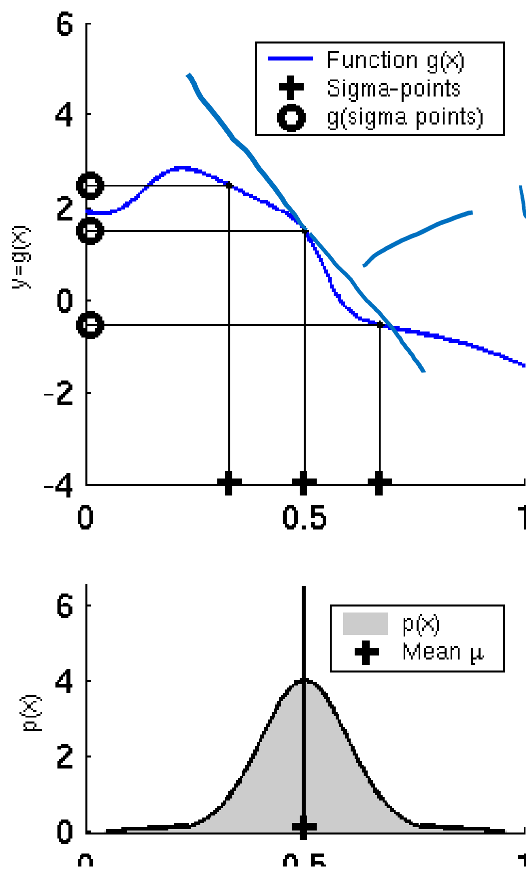
Unscented Kalman Filter

Linearization via Unscented Transform





EKF linearizes g(x)

EKF   UKF

neither filter can reproduce
the correct answer (gray dist)

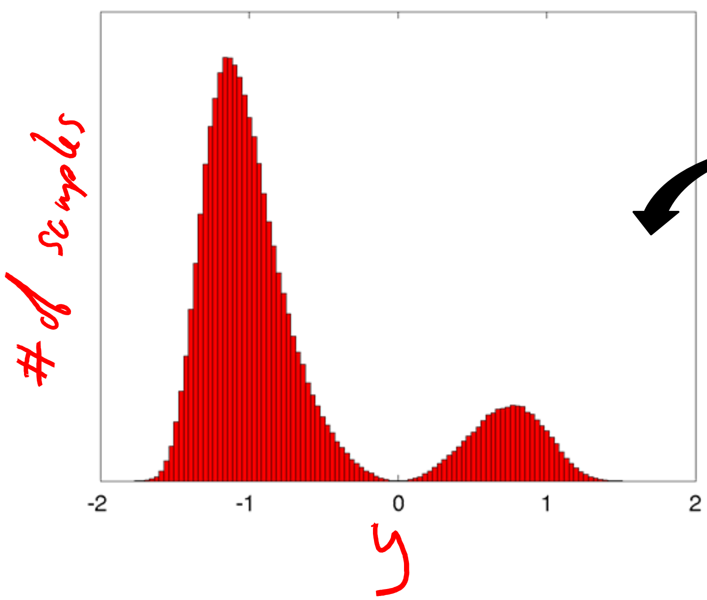
Unscented Transform

- intuition: it should be easier to approximate a given distribution than it is to approximate an arbitrary non-linear function
  - it is easy to transform a point through a non-linear function
  - use a set of points that capture the mean and covariance of the distribution, transform the points through the non-linear function, then compute the (weighted) mean and covariance of the transformed points

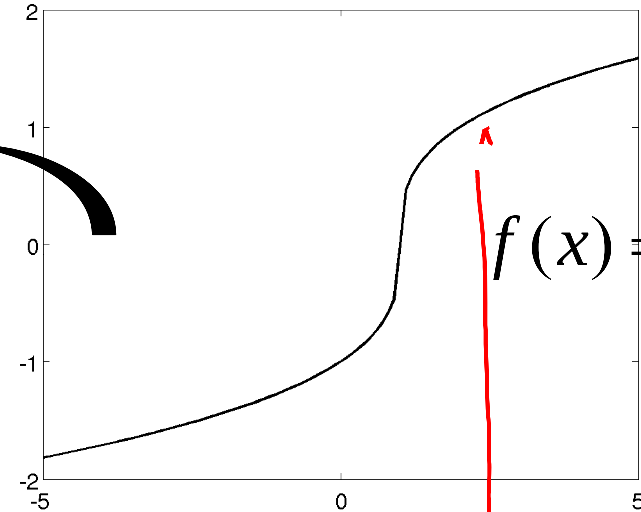
Empirical transformation of a Gaussian random variable

```
% generate 500,000 samples from N(0, 1)
x = randn(1, 500000);

% draw the histogram of x
xc = -5:0.2:5;
nx = hist(x, xc);
bar(xc, nx, 1);

% transform each sample by f(x)
y = nthroot(x – 1, 3);

% draw the histogram of y
xc = -2:0.1:2;
ny = hist(y, xc);
bar(xc, ny, 1);
```

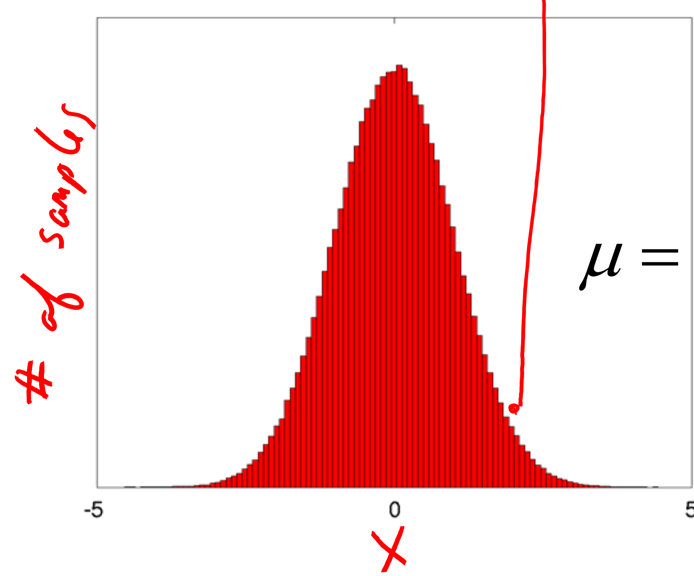
Empirical transformation of a Gaussian random variable



-transform 500,000 random samples through a non-linear function

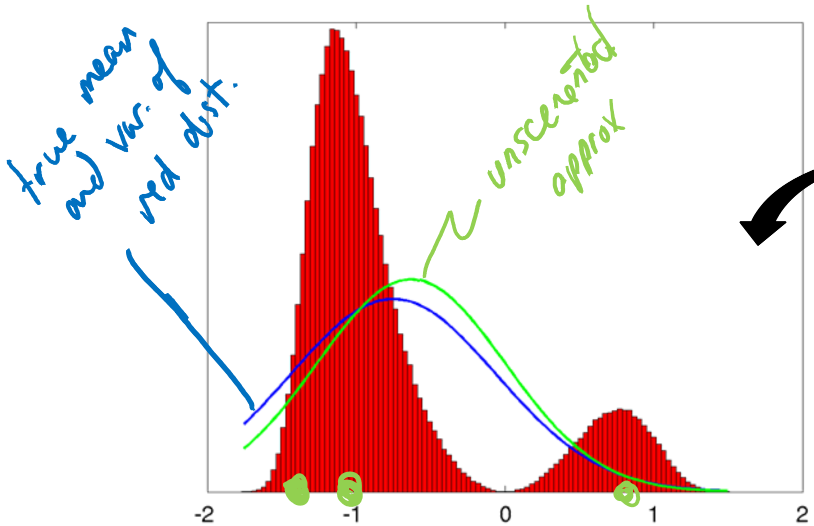


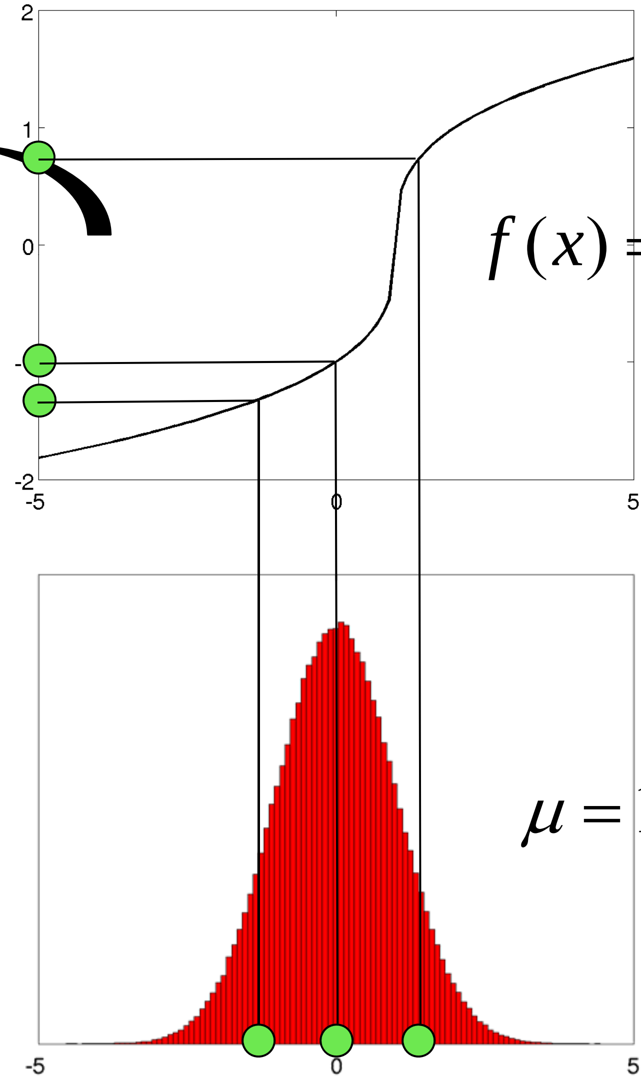
$$f(x) = \sqrt[3]{x-1}$$



$$\mu = 1, \Sigma = 1$$

Unscented Transform





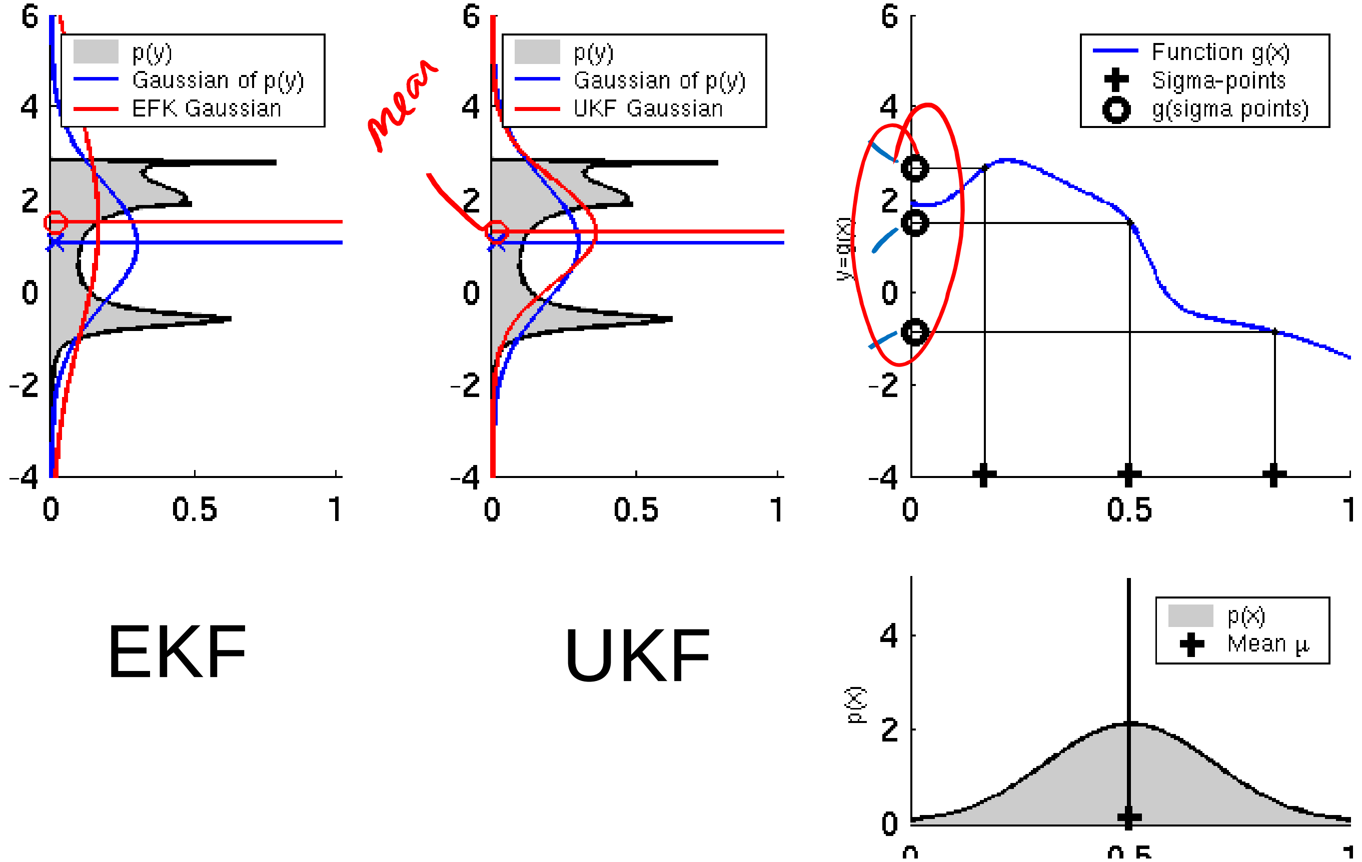
$$f(x) = \sqrt[3]{x-1}$$

$$\mu = 1, \Sigma = 1$$

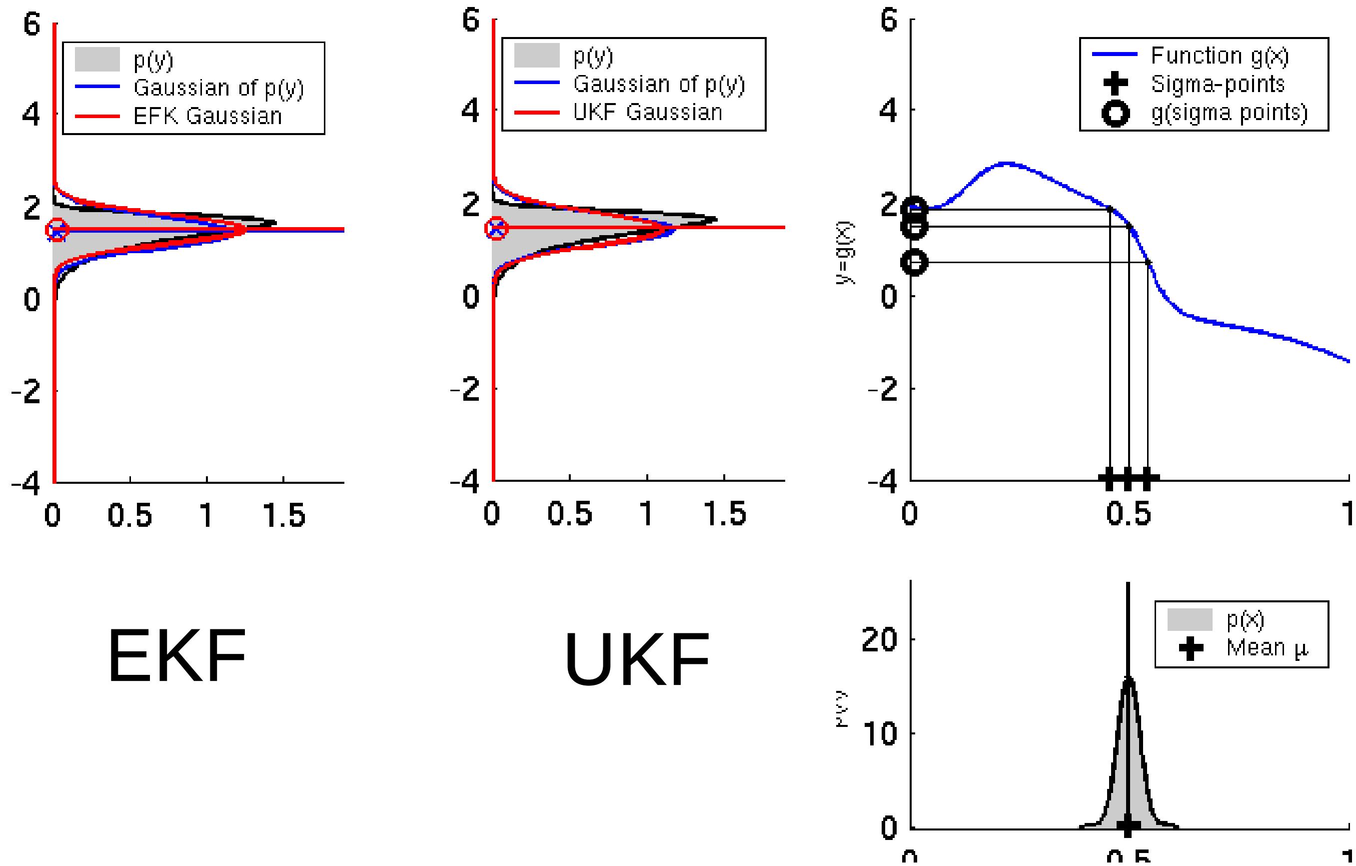
-transform 3 carefully chosen samples through a non-linear function

-samples are called sigma points

UKF Sigma-Point Estimate (2)



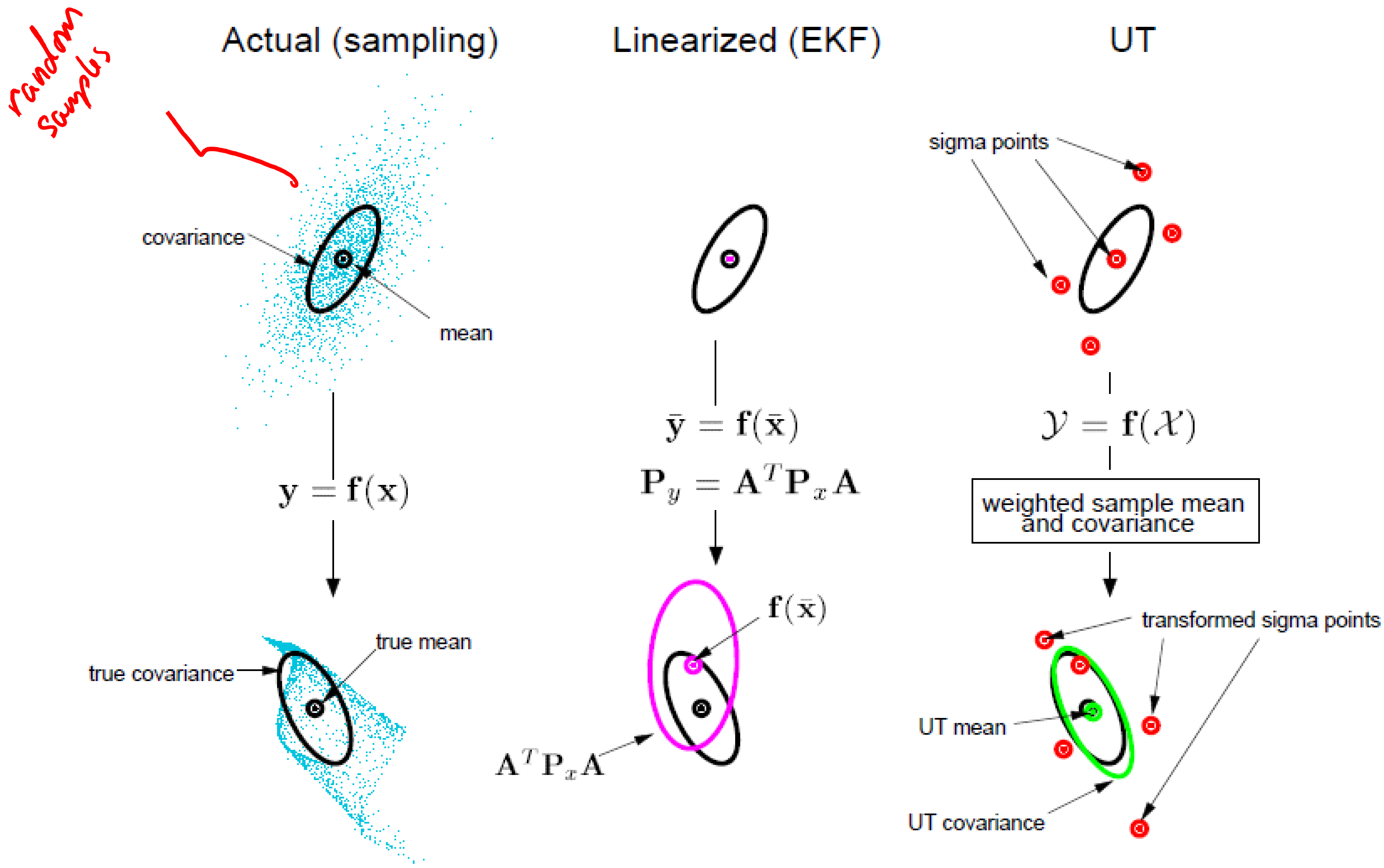
UKF Sigma-Point Estimate (3)



EKF

UKF

UKF Sigma-Point Estimate (4)



Unscented Transform

- for an n-dimensional Gaussian with mean μ and covariance Σ , the unscented transform uses 2n+1 sigma points (and associated weights)

2n+1 samples

Sigma points

$$\chi^0 = \mu$$

$$\chi^i = \mu \pm \left(\sqrt{(n+\lambda)\Sigma}\right)_i$$

matrix square root

tuning parameter

$$\lambda = \alpha^2 (n+\kappa) - n$$

Weights

$$w_m^0 = \frac{\lambda}{n+\lambda} \qquad w_c^0 = \frac{\lambda}{n+\lambda} + (1-\alpha^2+\beta)$$

$$w_m^i = w_c^i = \frac{1}{2(n+\lambda)} \qquad \text{for } i = 1,\ldots,2n$$

Unscented Transform

$$\lambda = \alpha^2 (n + \kappa) - n$$

- choose $\kappa \geq 0$ to guarantee a “reasonable” covariance matrix
  - value is not critical, so choose $\kappa = 0$ by default
- choose $0 \leq \alpha \leq 1$
  - controls the spread of the sigma point distribution; should be small when nonlinearities are strong
- choose $\beta \geq 0$
  - $\beta = 2$ is optimal if distribution is Gaussian

Unscented Transform

Pass sigma points through nonlinear function

$$\psi^i = g(\chi^i)$$

Recover mean and covariance

$$\mu' = \sum_{i=0}^{2n} w_m^i \psi^i$$

$$\Sigma' = \sum_{i=0}^{2n} w_c^i (\psi^i - \mu)(\psi^i - \mu)^T$$

UKF_localization (μ_{t-1}, Σ_{t-1}, u_t, z_t, m):

$X_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$, $n=3$

Prediction:

$$R_t = \begin{pmatrix} \left(\alpha_1 v_t^2 + \alpha_2 \omega_t^2\right)^2 & 0 \\ 0 & \left(\alpha_3 v_t^2 + \alpha_4 \omega_t^2\right)^2 \end{pmatrix}$$ Control noise

$$Q_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$ Measurement noise

* see "Augmented state" slides

state covariance

$$\mu_{t-1}^a = \begin{pmatrix} \mu_{t-1}^T & (0\;0)^T & (0\;0)^T \end{pmatrix}$$ Augmented state mean

$$\Sigma_{t-1}^a = \begin{pmatrix} \Sigma_{t-1} & 0 & 0 \\ 0 & R_t & 0 \\ 0 & 0 & Q_t \end{pmatrix}$$ Augmented covariance

* $$\chi_{t-1}^a = \begin{pmatrix} \mu_{t-1}^a & \mu_{t-1}^a + \gamma\sqrt{\Sigma_{t-1}^a} & \mu_{t-1}^a - \gamma\sqrt{\Sigma_{t-1}^a} \end{pmatrix}$$ Sigma points

$$\bar{\chi}_t^x = g\left(u_t + \chi_t^u, \chi_{t-1}^x\right)$$ Prediction of sigma points

$$\bar{\mu}_t = \sum_{i=0}^{2L} w_m^i \, \chi_{i,t}^x$$ Predicted mean

$$\bar{\Sigma}_t = \sum_{i=0}^{2L} w_c^i \left(\chi_{i,t}^x - \bar{\mu}_t\right)\left(\chi_{i,t}^x - \bar{\mu}_t\right)^T$$ Predicted covariance

UKF_localization (μ_{t-1}, Σ_{t-1}, u_t, z_t, m):

Correction:

measurement model

$$\bar{Z}_t = h(\bar{\chi}_t^x) + \chi_t^z$$ Measurement sigma points

$$\hat{z}_t = \sum_{i=0}^{2L} w_m^i \, \bar{Z}_{i,t}$$ Predicted measurement mean

$$S_t = \sum_{i=0}^{2L} w_c^i \, (\bar{Z}_{i,t} - \hat{z}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$ Pred. measurement covariance

$$\Sigma_t^{x,z} = \sum_{i=0}^{2L} w_c^i \, (\bar{\chi}_{i,t}^x - \bar{\mu}_t)(\bar{Z}_{i,t} - \hat{z}_t)^T$$ Cross-covariance

$$K_t = \Sigma_t^{x,z} S_t^{-1}$$ Kalman gain

$$\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$$ Updated mean

$$\Sigma_t = \bar{\Sigma}_t - K_t S_t K_t^T$$ Updated covariance

Augmented state

- to account for the control noise sigma points are generated for the control variables
- to account for the measurement noise sigma points are generated for the measurement variables
- all sigma points can be generated in a single step by using an augmented state vector

Augmented state

- Augmented state vector

$$\mu_{t-1}^{a} = \begin{bmatrix} \mu_{t-1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for additive control noise

for additive measurement noise

Augmented state sigma points

- Generating sigma points with the augmented state yields sigma points each having components in state, control, and measurement

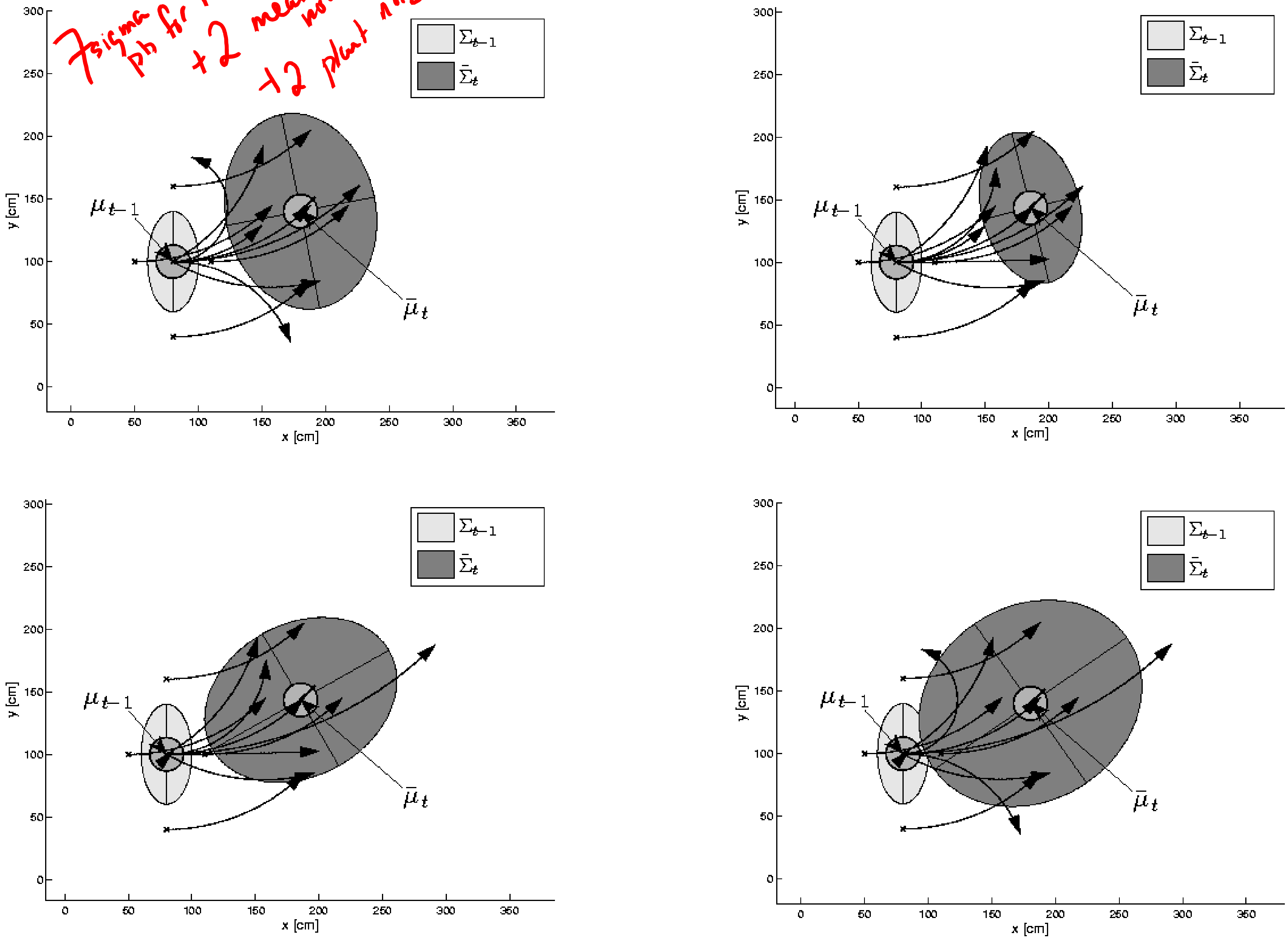
Augmented state sigma points

$$\chi_{t-1}^{a} = \begin{bmatrix} \chi_{t-1}^{x} \\ \chi_{t}^{\mu} \\ \chi_{t}^{z} \end{bmatrix} \begin{array}{l} \text{— state component of sigma points} \\ \text{— control component of sigma points} \\ \text{— measurement component of sigma points} \end{array}$$

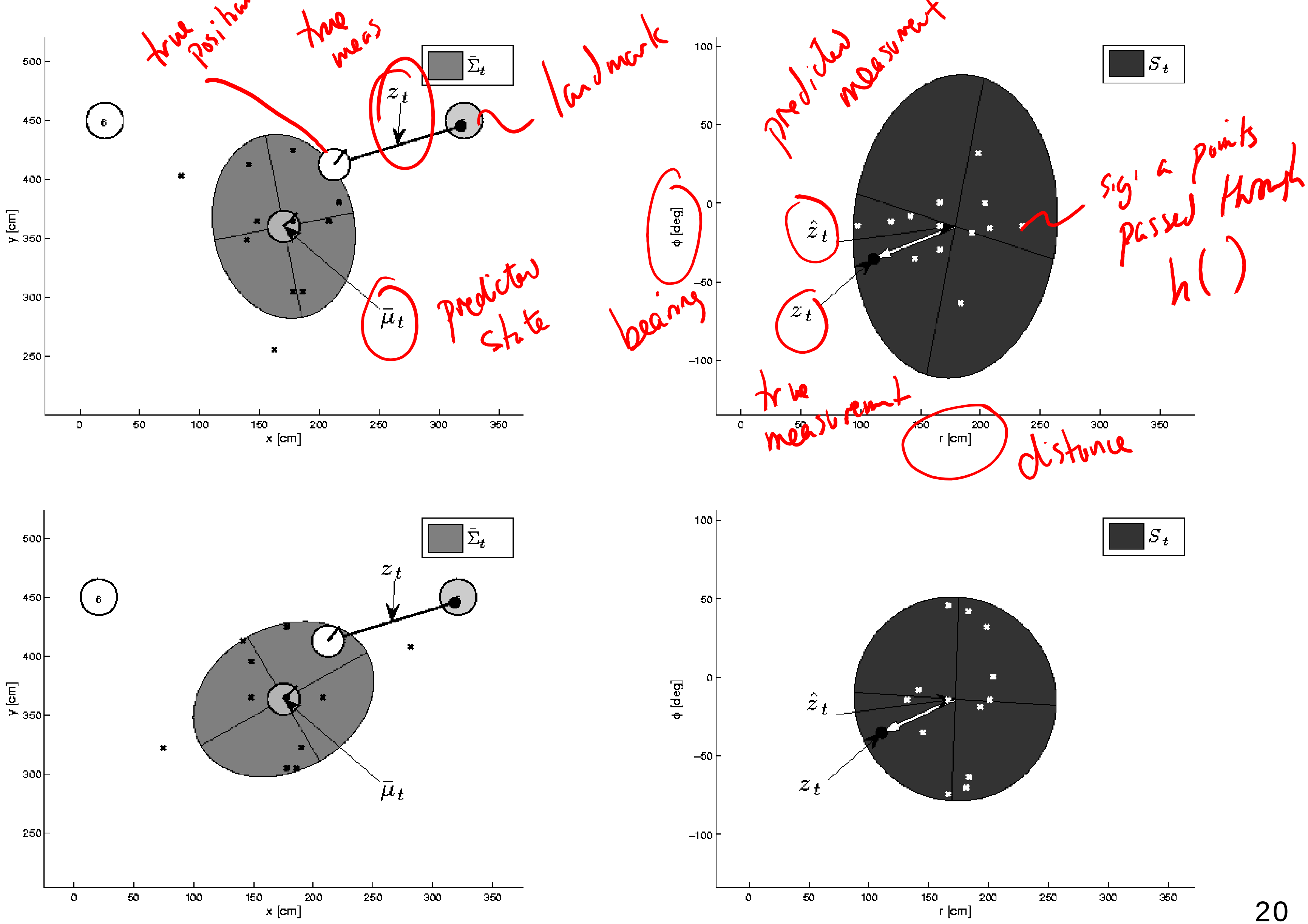
UKF Prediction Step

Note: σ not shown

7 sigma pts for μ +2 meas noise +2 plant noise

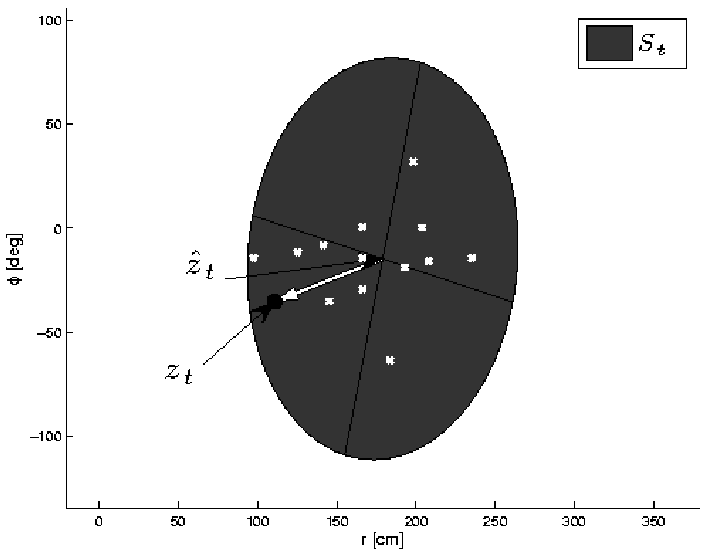


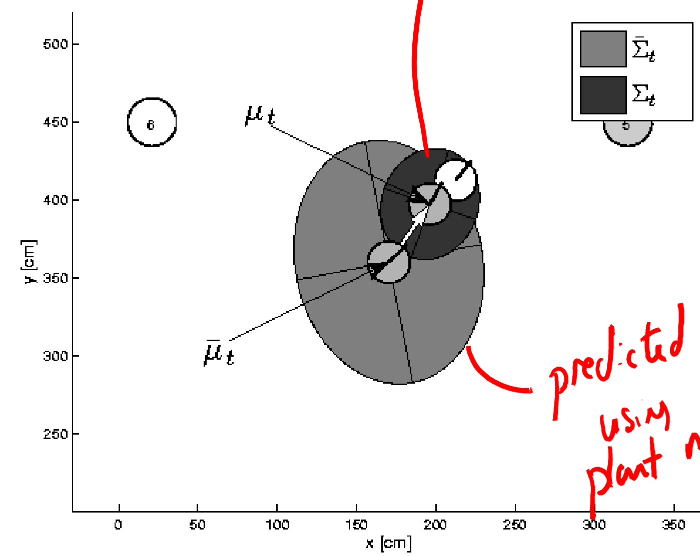
UKF Observation Prediction Step



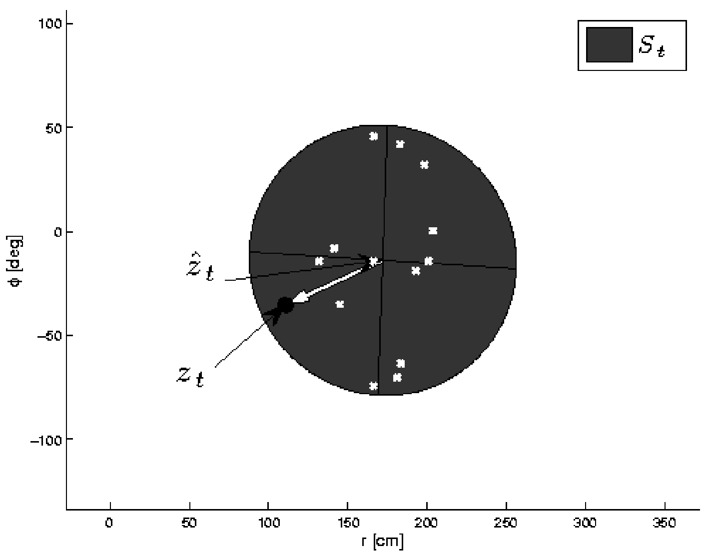
UKF Correction Step

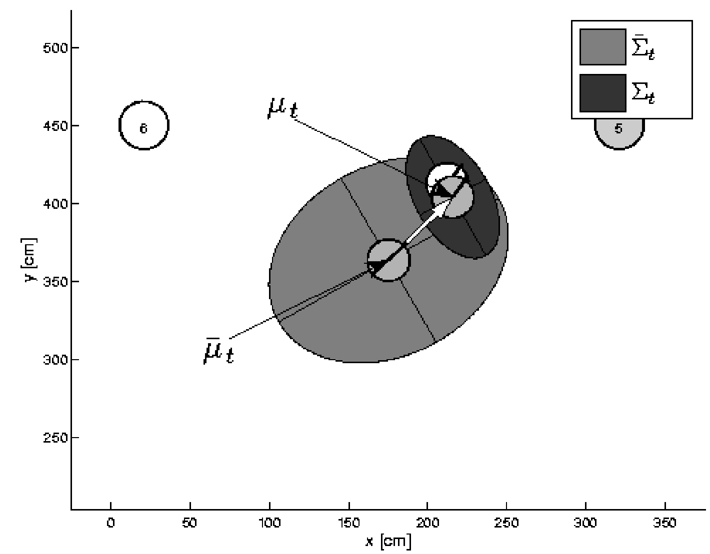
μ_t, Σ_t after incorporating z_t





predicted state using plant model

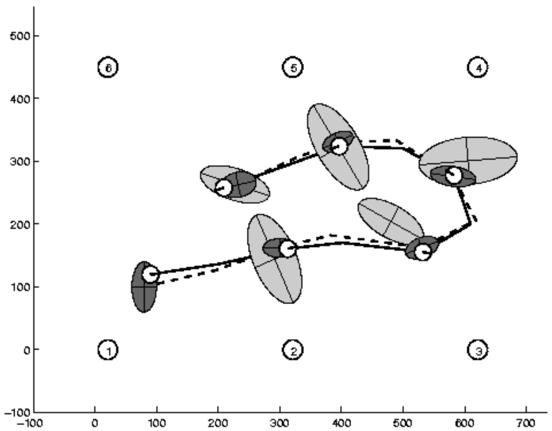




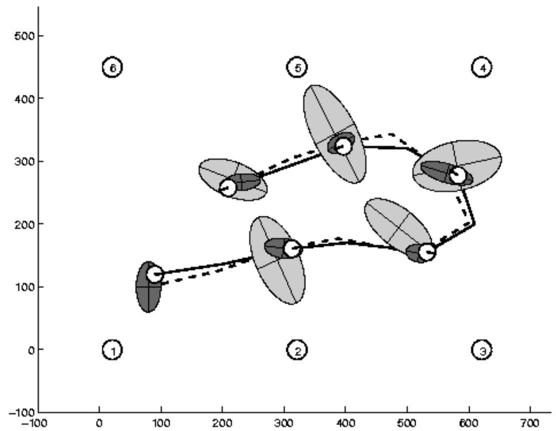
Estimation Sequence

predicted state
using plant
model

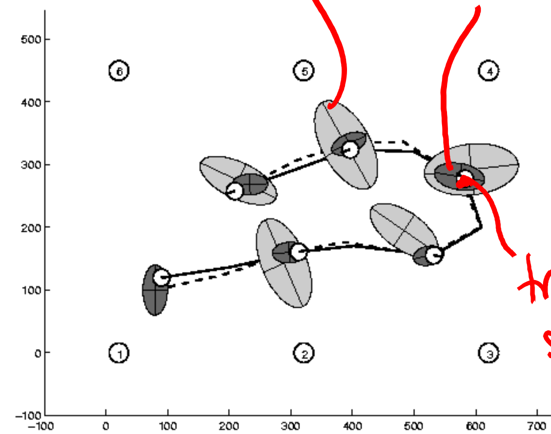
UKF estimated
state



EKF



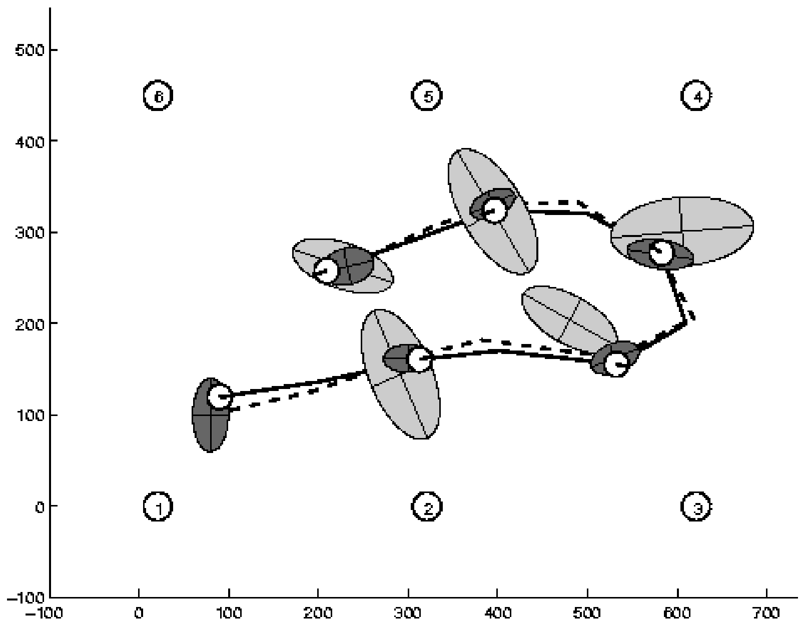
PF



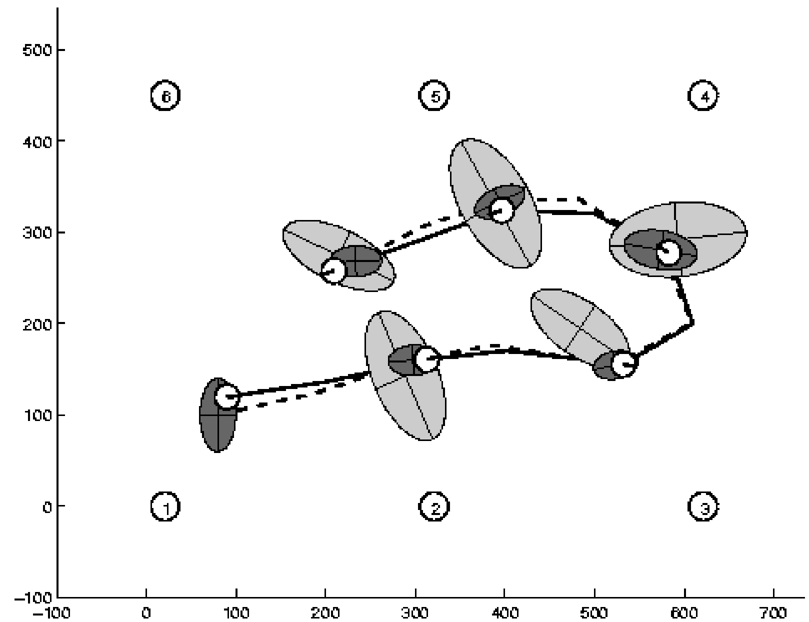
UKF

true
state

Estimation Sequence



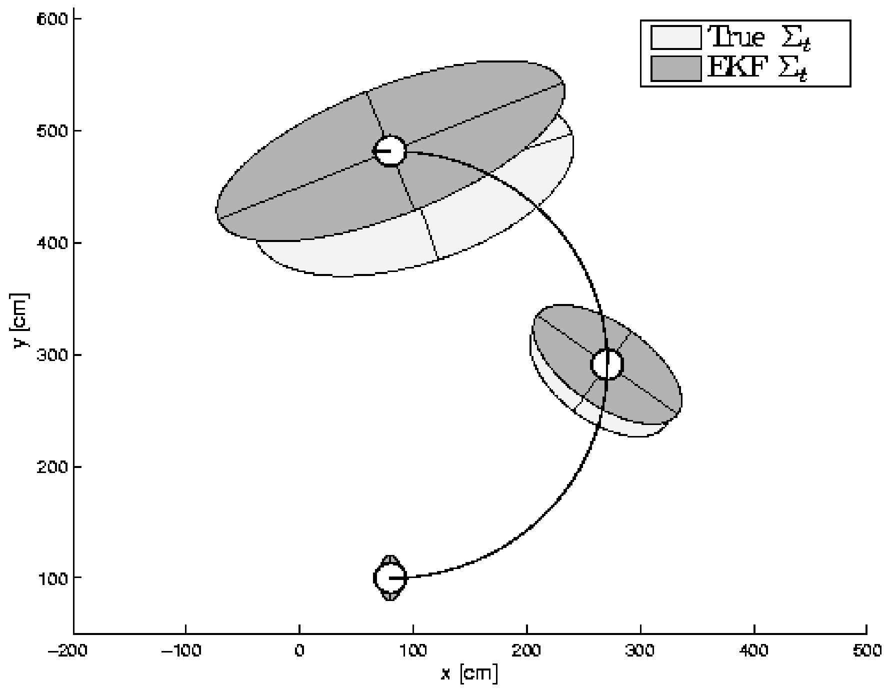
EKF



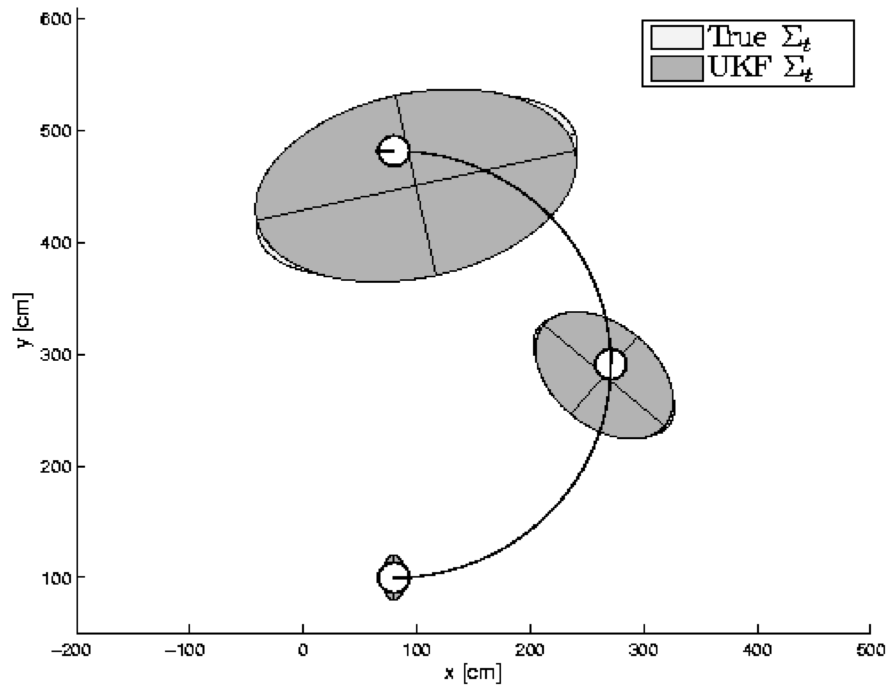
UKF

Prediction Quality

velocity_motion_model



EKF



UKF

UKF Summary

- Highly efficient: Same complexity as EKF, with a constant factor slower in typical practical applications
- Better linearization than EKF: Accurate in first two terms of Taylor expansion (EKF only first term)
- Derivative-free: No Jacobians needed
- Still not optimal!